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Yilmaz' theory of gravitation and some modifications

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Abstract. An apparent error in the formulation of Yilmaz' theory of gravitation is corrected. Yilmaz' field equations are unaffected but the corrected theory suggests alternative field equations.

1. Introduction

Yilmaz (1958) has suggested a new approach to general relativity based on a covariant generalization of the theory of a massless scalar field ϕ. Hoffmann (1960) has drawn attention to some difficulties in this theory and discussed certain modifications, and Dowker (1965) introduced a scalar theory of gravitation based on ideas similar to those of Yilmaz. In both these cases the authors start by considering the energy tensor of the scalar field. This energy tensor is correct, but Yilmaz' derivation of it from the Lagrangian density contains an error.

In this article we illustrate Yilmaz' error and some of its consequences. We shall also consider the results of alternative field equations to those used by Yilmaz.

The summation convention is used throughout. Greek suffixes take the values (0, 1, 2, 3) and Latin suffixes the values (1, 2, 3).

2. Yilmaz' theory

Yilmaz considers a space–time continuum

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

and supposes that a massless scalar field $\phi(x)$ is defined in this continuum. He seeks a solution of his field equations such that the $g_{\mu\nu}$ are functions of ϕ only, i.e. in the case of a static spherically symmetric space–time he requires the coordinates to be chosen so that the solution is of the form

$$\mathrm{d}s^2 = \mathrm{e}^{2\lambda}\,\mathrm{d}t^2 - \mathrm{e}^{2\sigma}(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2) \tag{1}$$

where λ, σ are functions of ϕ only.

The equations of motion of the field are obtained from the principle of stationary action

$$\delta\int L(\phi, \phi_\mu)(-g)^{1/2}\,\mathrm{d}^4x = 0 \tag{2}$$

where L is a scalar function of position and ϕ, $\phi_\mu = \partial\phi/\partial x^\mu$ are the quantities to be varied. Yilmaz obtains as the equation of motion

$$\frac{\partial L}{\partial\phi} - \left(\frac{\partial L}{\partial\phi_\mu}\right)_{;\mu} = 0 \tag{3}$$

where the semicolon denotes covariant differentiation with respect to x^μ. This equation is obtained by assuming that, for the purpose of the variation, the $(-g)^{1/2}$ in the integral (2) is independent of ϕ, although there seems little doubt that Yilmaz intended to impose his fundamental requirement $g_{\mu\nu} = g_{\mu\nu}(\phi)$ before the variation.

There is also another serious difficulty with equation (3). The quantity $\partial L/\partial\phi$ is usually calculated by varying ϕ and keeping ϕ_μ constant, but this is not a covariant procedure. Consequently in this conventional approach $\partial L/\partial\phi$ is not a scalar, which raises

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problems since the second term of (3) is undoubtedly a scalar. This difficulty can be overcome by introducing a covariant procedure for calculating $\partial L/\partial\phi$ such as by varying ϕ subject to $\phi_{\mu;\nu} = 0$. In this case $\partial L/\partial\phi$ is a true scalar and (3) is correct, but there is no evidence to suggest that Yilmaz intended to use other than the conventional procedure, so we shall take the view that (3) is incorrect.

The usual stress–energy tensor for a scalar field, viz.

$$T_{\mu}{}^{\nu} = \phi_{\mu}\frac{\partial L}{\partial \phi_{\nu}} - \delta^{\nu}_{\pi} L \tag{4}$$

is obtained, and from (3) and (4) it follows that this tensor has vanishing covariant divergence, i.e.

$$T^{\nu}_{\mu;\nu} = 0.$$

In order to find the actual form of L for the theory, Yilmaz proposes that the equation of motion (3) should be identical with the d'Alembert equation

$$\phi^{\mu}{}_{;\mu} = 0 \tag{5}$$

and asserts that a Lagrangian satisfying this requirement is

$$L = \frac{1}{8\pi}\phi_{\mu}\phi^{\mu} = \frac{1}{8\pi}g^{\mu\nu}\phi_{\mu}\phi_{\nu}. \tag{6}$$

However, if this expression for L is substituted into (3), the equation of motion is identical with (5) if, and only if,

$$\frac{\partial L}{\partial \phi} = 0. \tag{7}$$

From (6) this implies that $g^{\mu\nu}$ is independent of ϕ, which is contrary to Yilmaz' fundamental assumption.

3. The corrected theory

We now investigate the result of taking into account the dependence of $(-g)^{1/2}$ on ϕ in the integral (2) when carrying out the variation. For this purpose we define the scalar density $\mathscr{L}(\phi, \phi_{\mu})$ by

$$\mathscr{L} = L(-g)^{1/2}$$

so that (2) becomes

$$\delta \int \mathscr{L}(\phi, \phi_{\mu})\, \mathrm{d}^4x = 0$$

which leads to the equation of motion

$$\frac{\partial \mathscr{L}}{\partial \phi} - \left(\frac{\partial \mathscr{L}}{\partial \phi_{\mu}}\right)_{,\mu} = 0 \tag{8}$$

where the comma denotes partial differentiation. The corresponding stress–energy tensor density is

$$\mathscr{T}^{\nu}_{\mu} = \phi_{\mu}\frac{\partial \mathscr{L}}{\partial \phi_{\nu}} - \delta^{\nu}_{\mu}\mathscr{L}. \tag{9}$$

From (8) and the relation

$$\mathscr{L}_{,\mu} = \frac{\partial \mathscr{L}}{\partial \phi}\phi_{\mu} + \frac{\partial \mathscr{L}}{\partial \phi_{\nu}}\phi_{\mu,\nu} \tag{10}$$

it follows that $\mathscr{T}^{\nu}_{\mu}$ has vanishing ordinary divergence, i.e.

$$\mathscr{T}^{\nu}_{\mu,\nu} = 0.$$

Turning our attention to equation (8), we note that, since $(-g)^{1/2}$ is independent of ϕ_μ, the second term of (8) is, apart from sign,

$$\left\{\frac{\partial\mathscr{L}}{\partial\phi_\mu}(-g)^{1/2}\right\}_{,\mu} = (-g)^{1/2}\left(\frac{\partial L}{\partial\phi_\mu}\right)_{;\mu}$$
$$= \frac{1}{4\pi}(-g)^{1/2}\phi^\mu{}_{;\mu}$$

if L is given by (6). Hence equation (8) is identical with equation (5), provided that

$$\frac{\partial\mathscr{L}}{\partial\phi} = 0. \tag{11}$$

Unlike equation (7), this does not imply that $g^{\mu\nu}$ is independent of ϕ. Since ϕ is time-independent, equation (11) is

$$\frac{\partial}{\partial\phi}\{(-g)^{1/2}(g^{11}\phi_1\phi_1+g^{22}\phi_2\phi_2+g^{33}\phi_3\phi_3)\} = 0$$

which, using the metric (1), leads to

$$\bar{\lambda}+\bar{\sigma} = 0 \tag{12}$$

where the bar denotes differentiation with respect to ϕ. This condition reduces (5) to

$$\nabla^2\phi = 0$$

so that ϕ can be taken to be the usual Newtonian gravitational potential, i.e.

$$\phi = \frac{m}{r} \tag{13}$$

where the units are chosen so that $c = G = 1$.

4. The field equations

Yilmaz' energy tensor (4) has vanishing covariant divergence, so he proposes the following field equations:

$$G_\mu^\nu \equiv R_\mu^\nu - \tfrac{1}{2}\delta_\mu^\nu R = 8\pi T_\mu^\nu \tag{14}$$

since the Einstein tensor G_μ also has vanishing covariant divergence.

From equation (9) we see that, since $(-g)^{1/2}$ is independent of ϕ_ν,

$$\mathscr{T}_\mu^\nu = \phi_\mu\frac{\partial L}{\partial\phi_\nu}(-g)^{1/2} - \delta_\mu^\nu L(-g)^{1/2}$$
$$= T_\mu^\nu(-g)^{1/2}$$

so that

$$\mathscr{T}_{\mu,\nu}^\nu \equiv \{(-g)^{1/2}T_\mu^\nu\}_{,\nu} = (-g)^{1/2}T_{\mu;\nu}^\nu + \Gamma_{\mu\sigma}^\nu T_\nu^\sigma(-g)^{1/2}. \tag{15}$$

Since the ordinary divergence of $\mathscr{T}_\mu^\nu$ vanishes, it follows that, in general, $T_{\mu;\nu}^\nu \neq 0$, so Yilmaz' field equations (14) will not hold unless the last term on the right-hand side of (15) is zero. We shall now show that for the Lagrangian (6) this term is zero:

$$\Gamma_{\mu\sigma}^\nu T_\nu^\sigma = \Gamma_{\mu\sigma}^\nu\left(\phi_\nu\frac{\partial L}{\partial\phi_\sigma} - \delta_\nu^\sigma L\right)$$
$$= \Gamma_{\mu\sigma}^\nu\phi_\nu\frac{\partial L}{\partial\phi_\sigma} - \Gamma_{\mu\nu}^\nu L. \tag{16}$$

For the Lagrangian (6) we have, from (10) and (11),

$$\mathscr{L}_{,\mu} = \frac{\partial \mathscr{L}}{\partial \phi_\nu}\phi_{\mu,\nu}$$
$$= \frac{\partial L}{\partial \phi_\nu}\phi_{\mu,\nu}(-g)^{1/2}$$

but also

$$\mathscr{L}_{,\mu} \equiv \{L(-g)^{1/2}\}_{,\mu} = L_{,\mu}(-g)^{1/2} + L\Gamma^{\rho}_{\rho\mu}(-g)^{1/2}$$

so that

$$L_{,\mu} = \frac{\partial L}{\partial \phi_\nu}\phi_{\mu,\nu} - L\Gamma^{\rho}_{\rho\mu}. \tag{17}$$

From equations (16) and (17) we find

$$\Gamma^{\nu}_{\mu\sigma}T^{\sigma}_{\nu} = \Gamma^{\nu}_{\mu\sigma}\phi_\nu \frac{\partial L}{\partial \phi_\sigma} + L_{,\mu} - \frac{\partial L}{\partial \phi_\sigma}\phi_{\sigma,\mu}$$
$$= L_{;\mu} - \frac{\partial L}{\partial \phi_\sigma}\phi_{\sigma;\mu}.$$

Hence

$$\mathscr{T}^{\nu}_{\mu,\nu} = 0 \Rightarrow T^{\nu}_{\mu;\nu} = 0$$

provided that

$$L_{;\mu} - \frac{\partial L}{\partial \phi_\sigma}\phi_{\sigma;\mu} = 0.$$

This condition is easily seen to be satisfied by the Lagrangian (6), so Yilmaz' field equations (14) are still valid for his choice of Lagrangian, despite the error in his equation of motion.

5. The three classical tests

Before discussing the solution of Yilmaz' field equations, we shall digress to note the relationship between certain four-dimensional spherically symmetric isotropic space–times and the three classical tests of general relativity: viz. the deflection of light, the gravitational red shift and the advance of the perihelion of Mercury, as predicted by the Schwarzschild solution.

It is easily shown (see Page and Tupper 1968) that for ϕ given by (13) the space–time

$$ds^2 = (1 + a_1\phi + a_2\phi^2)\,dt^2 - (1 + b_1\phi)(dx^2 + dy^2 + dz^2)$$

gives the following values (Schwarzschild values taken as unity) for the three tests:

$$\left.\begin{array}{l} \text{deflection of light rays: } \tfrac{1}{4}(b_1 - a_1) \\ \text{gravitational red shift: } -\tfrac{1}{2}a_1 \\ \text{advance of perihelion: } \tfrac{1}{6}(a_1{}^2 - a_2 - a_1 b_1) \end{array}\right\}. \tag{18}$$

In addition, we must have $a_1 < 0$ for the central mass to attract the test particle and $a_1 = -2$ for the weak-field approximation to correspond to the Newtonian theory.

From (18) we see that any space–time with a metric which can be expanded in ascending powers of $\phi = m/r$ in the form

$$ds^2 = (1 - 2\phi + 2\phi^2)\,dt^2 - (1 + 2\phi)(dx^2 + dy^2 + dz^2) \tag{19}$$

will give the Schwarzschild values for the three tests and will also satisfy the additional conditions.

6. Yilmaz' solution

The solution of Yilmaz' field equations (14) for the metric (1) and for T_μ^ν given by (4) is found to be

$$ds^2 = e^{-2\phi}dt^2 - e^{2\phi}(dx^2 + dy^2 + dz^2). \tag{20}$$

This solution is found from the field equations by using equations (5) and (12). The ten field equations reduce to the following three equations for σ:

When $(\mu, \nu) = (i, j), i = j$: $\quad {}^{-2} = 1$

$(\mu, \nu) = (i, j), i \neq j$: $\quad \bar{\sigma}^2 = 1$

$(\mu, \nu) = (0, 0)$: $\quad 2\bar{\bar{\sigma}} + \bar{\sigma}^2 = 1.$

When $(\mu, \nu) = (0, i)$ or $(i, 0)$, the equations are identically zero.

From the equations we see that

$$\sigma = \pm\phi + \text{constant}$$

and the final choice $\sigma = \phi$, giving the metric (20), is governed by (19), so that the three tests and the additional conditions are satisfied.

The remarkable feature of Yilmaz' field equations is that they are ten equations for only one unknown scalar function, and yet they lead to a solution.

It should be noted that, although Yilmaz gives only (20) as the solution to his field equations, it is not clear that this is the only acceptable solution since, in his formulation of the theory, the condition (12) does not arise. The corrected theory has the advantage of reducing the solutions to the two possibilities $\sigma = -\lambda = \pm\phi$.

7. The scalar field equation

For a scalar theory it is usual to have one scalar field equation, rather than ten equations as in Yilmaz' theory (e.g. see Trautman 1965, p. 149). We now investigate the consequences of replacing (14) by the single equation

$$G = 8\pi k T \tag{21}$$

where $G = G_\mu^\mu$, $T = T_\mu^\mu$ and we have introduced a coupling constant k. It should be noted that, if k had been introduced into the field equations of the previous section, it would necessarily have taken unit value if the Schwarzschild values for the three tests were to be obtained.

Equation (21) leads to the equation

$$\bar{\bar{\sigma}} + \bar{\sigma}^2 = k. \tag{22}$$

When $k = n^2 > 0$ this equation has the solution

$$e^\sigma = \cosh n\phi + A \sinh n\phi$$

where A is a constant. By expanding this expression in ascending powers of ϕ we find

$$e^{2\sigma} = 1 + 2An\phi$$
$$e^{-2\sigma} = 1 - 2An\phi + (3A^2 - 1)n^2\phi^2. \tag{23}$$

From these expressions we see that, apart from the advance of the perihelion, the Schwarzschild values are obtained and the other conditions satisfied if we take

$$An = 1.$$

Then we have

$$e^\sigma = \cosh n\phi + \frac{1}{n}\sinh n\phi$$

and, to the required power of ϕ,

$$e^{2\sigma} = 1+2\phi$$
$$e^{-2\sigma} = 1-2\phi+(3-n^2)\phi.$$

From (18) the advance of the perihelion is $(5+n^2)/6$ of the Schwarzschild value. Hence, for $k = n^2 > 0$, the space–time

$$ds^2 = \left(\cosh n\phi + \frac{1}{n}\sinh n\phi\right)^{-2} dt^2 - \left(\cosh n\phi + \frac{1}{n}\sinh n\phi\right)^{2} (dx^2+dy^2+dz^2)$$

satisfies the scalar field equation (21) and gives the Schwarzschild values for the deflection of light rays and the gravitational red shift. For the advance of the perihelion the value will be more than $\frac{5}{6}$ of the Schwarzschild value and will be near the Schwarzschild value when k is near unity.

When $k = n^2 = 1$, the metric is of Yilmaz' form (20) and so satisfies the three tests.

The case $k < 0$ may be disregarded since the solution of equation (22) is then in terms of circular functions.

8. Alternative field equations

The fact that the equation of motion (13) leads to the stress–energy tensor density $\mathscr{T}_\mu^\nu$, with vanishing ordinary divergence, suggests that we could form field equations for the theory by equating $\mathscr{T}_\mu^\nu$ with one of the stress–energy complexes used in general relativity. We shall consider here two such field equations, using, respectively, the Einstein and Møller complexes given by (Trautman 1962, p. 190):

$${}_{\mathrm{E}}\theta_\mu^\nu = U_\mu^{\nu\lambda}{}_{,\lambda} = \left[\frac{1}{8\pi}(-g)^{1/2}g_{\mu\alpha}\{g(g^{\beta\lambda}g^{\nu\alpha}-g^{\nu\beta}g^{\lambda\alpha})\}_{,\beta}\right]_{,\lambda}$$

where $U_\mu^{\nu\lambda}$ is the 'superpotential' of von Freud (1939), and

$${}_{\mathrm{M}}\theta_\mu^\nu = {}_{\mathrm{M}}U_\mu^{\nu\lambda}{}_{,\lambda} = \left\{\frac{1}{8\pi(-g)^{1/2}}g^{\nu\alpha}g^{\lambda\beta}(g_{\mu\beta,\alpha}-g_{\mu\alpha,\beta})\right\}_{,\lambda}.$$

Under linear transformations both complexes transform like tensor densities of weight 1, unlike the Landau–Lifshitz complex which transforms like a tensor density of weight 2 (Trautman 1962, p. 191). Since $\mathscr{T}_\mu^\nu$ is a tensor density of weight 1, we shall attempt to form field equations by equating $\mathscr{T}_\mu^\nu$ with ${}_{\mathrm{E}}\theta_\mu^\nu$ or ${}_{\mathrm{M}}\theta_\mu^\nu$ only. It should be noted that both complexes are antisymmetric in ν and λ, and so have vanishing divergence, in common with $\mathscr{T}_\mu^\nu$.

We are thus led to suggest the following possible field equations:

$${}_{\mathrm{E}}\theta_\mu^\nu = \mathscr{T}_\mu^\nu \tag{24}$$

or

$${}_{\mathrm{M}}\theta_\mu^\nu = \mathscr{T}_\mu^\nu. \tag{25}$$

However, it is easily shown that, with the metric (1) and the Lagrangian (6), neither of these equations has a solution. This is disappointing, but not particularly surprising, since they each form a set of ten equations for effectively one unknown quantity, viz. the σ (or λ) in the metric (1).

Bearing in mind the results of the previous section, we suggest that equations (24) and (25) should be replaced by the contracted equations

$${}_{\mathrm{E}}\theta = k\mathscr{T} \tag{26}$$

and

$${}_{\mathrm{M}}\theta = k\mathscr{T} \tag{27}$$

where ${}_{\mathrm{E}}\theta = {}_{\mathrm{E}}\theta_\mu^\mu$, ${}_{\mathrm{M}}\theta = {}_{\mathrm{M}}\theta_\mu^\mu$, $\mathscr{T} = \mathscr{T}_\mu^\mu$ and k is a coupling constant.

Using equations (5) and (12) we find that (26) reduces to

$$\bar{\bar{\sigma}} = k \tag{28}$$

i.e.

$$\sigma = \tfrac{1}{2}k\phi^2 + \alpha\phi$$

where α is a constant and we have used the condition $\sigma \to 0$ as $\phi \to 0$. Hence, to the second order in ϕ,

$$e^{2\sigma} = \exp(k\phi^2 + 2\alpha\phi) = 1 + 2\alpha\phi$$

$$e^{-2\sigma} = \exp(-k\phi^2 - 2\alpha\phi) = 1 - 2\alpha\phi + (2\alpha^2 - k)\phi^2.$$

For the weak-field approximation to agree with Newtonian theory, we must have $\alpha = 1$. We then find the following values for the three tests, as compared with the Schwarzschild values:

advance of the perihelion: $1 + \frac{1}{6}k$

deflection of light rays: 1

gravitational red shift: 1.

Since k can take any non-zero value, the field equation (26) will give good agreement with the Schwarzschild values if we make k as small as we please. If we do not choose α equal to unity, then, at the cost of exact agreement between Newtonian and weak-field theory, the values for the three tests become $\alpha^2 + \frac{1}{6}k$, α and α, respectively. Dicke (1963) has made the point that the experimental evidence agrees with the predictions of general relativity only to an accuracy of between 5 and 20%, so, if $\alpha \neq 1$ and k is chosen suitably, this theory may agree with the experimental evidence.

If we use (27) as our field equation instead of (26), we find

$$3\,\bar{\bar{\sigma}} = k.$$

Since k can be chosen to have any non-zero value, this equation is effectively the same as equation (28). It follows that the field equations (26) and (27) lead to identical results.

9. Zero coupling constant theories

It will be noticed that the field equations of the last section give exact agreement with general relativity for the three tests if we put $\alpha = 1$ and $k = 0$. This suggests that it may be interesting to consider the field equations of §§ 7 and 8 with a zero coupling constant. Of course, the field equations will no longer contain the energy tensor (or energy tensor density) of the scalar field and such theories will have no connection with the original Yilmaz formulation. The physical basis, if any, of these theories is obscure, but they give an interesting variation on the theme of scalar gravitational theories and may be worth noting, if only for their curiosity value.

The theories of Nordström (1913) and Littlewood (1953) are equivalent to the field equations

$$C^{\mu}_{\rho\nu\sigma} = 0 \quad \text{and} \quad R = 0$$

(see Schild 1962, p. 112). Instead of the field equation $C^{\mu}_{\rho\nu\sigma} = 0$, we use the de Donder harmonic coordinate condition

$$\{(-g)^{1/2}g^{\mu\nu}\}_{,\nu} = 0 \tag{29}$$

applied to a static spherically symmetric isotropic space–time. This leads to a metric of the form

$$ds^2 = e^{-2\sigma}\,dt^2 - e^{2\sigma}(dx^2 + dy^2 + dz^2)$$

where σ is a function of the radial coordinate r only, and so can be regarded as a function of the potential $\phi = m/r$.

For the field equation to be used in connection with this space–time we choose the same equation $R = 0$ (or $G = 0$) as in the Nordström theory. The resulting equations are then those of § 7 with $k = 0$. The equation $R = 0$ gives

$$\bar{\bar{\sigma}}+\bar{\sigma}^2 = 0$$

so that

$$e^\sigma = \alpha\phi+\beta.$$

By choosing $\alpha = \beta = 1$, we obtain the space–time with metric

$$ds^2 = (1+\phi)^{-2}\,dt^2-(1+\phi)^2(dx^2+dy^2+dz^2).$$

This leads to the Schwarzschild values for light-ray deflection and the gravitational red shift. The value for the advance of the perihelion is $\frac{5}{6}$ of the Schwarzschild value, as expected by putting $k = n = 0$ in the results of § 7.

If, instead of the equation $R = 0$, we use the equations

$$_{\mathrm{E}}\theta = 0 \quad \text{or} \quad _{\mathrm{M}}\theta = 0 \tag{30}$$

where $_{\mathrm{E}}\theta$, $_{\mathrm{M}}\theta$ are defined in § 8, we find

$$\sigma = \alpha\phi.$$

By choosing $\alpha = 1$ we obtain the Yilmaz form for space–time, so that the three tests are satisfied.

The equations (29) and (30) are not tensor equations and are invariant only under linear transformations of the coordinates. This invariance is no worse than that in Yilmaz' basic formulation of his theory, although Yilmaz' field equations are covariant.

10. Conclusion

The error in Yilmaz' formulation of his theory has been corrected, resulting in the possibility of various tensor density field equations. Unlike Yilmaz' field equations, none of the alternatives discussed here gives exact agreement with general relativity, but they give values which are within the limits of observational evidence.

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